[ENG30]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

B.Sc. (Hons) MATHEMATICS

SEMESTER 1 EXAMINATIONS 2021/22

COMPLEX VARIABLES

MODULE NO: MMA6006

Date: Thursday 13th January 2022 **Time: 10.00 – 12.15**

INSTRUCTIONS TO CANDIDATES:

1. **Answer ALL FOUR questions.**
2. **All questions carry EQUAL marks.**
3. **Maximum marks for each part/question are shown in brackets.**

---

1. (a) (i) The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $u(x, y)$ and $v(x, y)$ satisfy $\nabla^2 u = \nabla^2 v = 0$ where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

(6 marks)

(ii) Show that

$$u(x, y) = x^2 - y^2 - 2xy - 2x + 3y$$

satisfies $\nabla^2 u = 0$. (4 marks)

(iii) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function and $u(x, y)$ is the function given in Question 1(a)(ii) above, find the function $v(x, y)$ given that $v(0,0) = 0$.

(7 marks)

(iv) Hence find the function $f(z) = u + iv$ in Question 1(a)(iii) above in a form in which the right-hand side is also written as a function of z.

(4 marks)

(b) Show that $f(z) = \bar{z}$, where $\bar{z}$ is the complex conjugate of z, is **not** an analytic function.

(4 marks)

2. Find the value of

$$\int_C z^2 dz$$

along the paths C described in each of the cases (a), (b) and (c) below, all of which refer to the diagram *Figure Q2*:

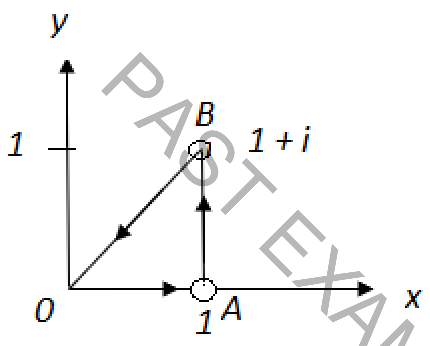

Figure Q2

(a) C is the line OB joining the points $z = 0$ to $z = 1 + i$;

(10 marks)

(b) C is the path OAB ;

(10 marks)

(c) C is the closed contour $OABO$, using your results for Questions 2(a) and (b) above **and** any other method.

(5 marks)

3. (a) If

$$f(z) = \frac{z}{(z+2)(z-2i)^3}$$

and *D*, *E* and *F* are circles with radius *3* and centres at $-2i, 0$ and $3i$ respectively, find:

(i) $\oint_D f(z)dz;$

(8 marks)

(ii) $\oint_E f(z)dz;$

(8 marks)

(iii) $\oint_F f(z)dz.$

(4 marks)

(b) State the value of

$$\oint_F \frac{dz}{(z-2i)^3}$$

where *F* is defined in part (a) above.

Hence, or otherwise, find

$$\oint_F \frac{2dz}{(z+2)(z-2i)^3}\ .$$

(5 marks)

4. (a) Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$$

by considering a suitable contour integral.

(10 marks)

(b) (i) The function $f(z)/g(z)$ has a simple pole at the point $z = a$. Use L'Hopital's rule to show that the residue at this point is

$$f(a)/g'(a).$$

(5 marks)

(ii) Show that the sum of residues on the negative real axis, including $z = 0$, of

$$\frac{e^z}{\sin mz}$$

is

$$\frac{e^{\pi/2m}}{2m} \operatorname{sech} \frac{\pi}{2m}$$

where m is a positive real number excluding zero.

(10 marks)

END OF QUESTIONS