UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

BSc (Hons) MATHEMATICS

SEMESTER 1 EXAMINATIONS 2021/22

FURTHER LINEAR ALGEBRA

MODULE NO: MMA6002

Date: Tuesday 11th January 2022 Time: 10:00 - 12:15

INSTRUCTIONS TO CANDIDATES:

1. **Answer all FOUR questions.**
2. **All questions carry equal marks.**
3. **Maximum marks for each part/question are shown in brackets.**

1. (a) Consider the set of 3×3 upper triangular real matrices:

$$ut(3,\boldsymbol{R}) = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a,b,c,d,e,f \in \boldsymbol{R} \right\}.$$

Show that $ut(3,\boldsymbol{R})$ is a subspace and a Lie subalgebra of the general linear Lie algebra $gl(3,\boldsymbol{R})$.

(8 marks)

(b) Explain what is meant by the *derived algebra* of a Lie algebra L.

Describe the derived algebra L' of the Lie algebra $ut(3,\boldsymbol{R})$ of part (a) above.

(5 marks)

(c) Show that

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

is a basis for $o(3)$.

(6 marks)

Hence show that $o(3)$ is isomorphic to the Lie algebra on $\boldsymbol{R}^3$ given by the cross product.

(6 marks)

PLEASE TURN THE PAGE....

2. (a) Let G be a group of complex $n \times n$ matrices and let $a > 0$ be a real number.

Consider the set S of all differentiable curves $\gamma : (-a, a) \to G$ satisfying $\gamma(0) = I$, where I is the identity matrix.

The *tangent space* of G is $TG = \{\gamma'(0) : \gamma \in S\}$.

Show that TG is a subspace of $Mat(n, \boldsymbol{C})$.

(12 marks)

(b) Explain carefully why the dimension of the real vector space $su(n)$ of skew-hermitian complex $n \times n$ matrices with zero trace is $n^2 - 1$.

Given that the dimension of the vector space $o(n)$ of skew-symmetric real matrices is $\frac{1}{2}n(n-1)$, find the dimension of the matrix groups $SU(23)$ and $SO(33)$.

(7 marks)

(c) Determine the centres of the groups $SU(23)$ and $SO(33)$.

State, with reasons, whether or not the groups $SU(23)$ and $SO(33)$ are isomorphic.

You may assume that all matrices in the centres are scalar matrices.

(6 marks)

PLEASE TURN THE PAGE....

3. (a) Let A be a normed real algebra with basis $\{e_1, e_2, ..., e_n\}$ where $e_1 = 1$.

Show that $e_i^2 = -1$ for all $i \geq 2$.

You may assume that $\overline{xy} = \bar{y}\,\bar{x}$ for all $x, y \in A$.

(8 marks)

(b) For the quaternions $q_1 = 5 + 2i + 3j - 4k$ and $q_2 = 3 - 6i + 4j - 2k$ calculate

(i) $q_1 + q_2$ (ii) $q_1 q_2$

(iii) $q_2 q_1$ (iv) q_1^{-1}

(9 marks)

(c) By identifying the quaternions $\boldsymbol{H}$ with $\boldsymbol{R}^4$ we may define a linear transformation $\boldsymbol{R}^4 \to \boldsymbol{R}^4$ by multiplication by a quaternion.

Find the matrix that represents the linear transformation $f : \boldsymbol{R}^4 \to \boldsymbol{R}^4$ given by $f(v) = vq$, where $q = 8 + 4i + 2j + k$.

(8 marks)

PLEASE TURN THE PAGE....

4. (a) Consider the subalgebras $B = Mat(3, \boldsymbol{R})$ and $C = \{qI : q \in \boldsymbol{H}\}$, where I is the 3×3 identity matrix, of $Mat(3, \boldsymbol{H})$ thought of as a *real* algebra.

Explain why $xy = yx$ for all $x \in B, y \in C$. (3 marks)

Let $A = \{\sum xy : x \in B, y \in C\}$.

Explain why $A = Mat(3, \boldsymbol{H})$. (4 marks)

Define a linear mapping $\phi : B \otimes C \to A$ by $\phi(x \otimes y) = xy$.

Show that ϕ is an algebra map. (2 marks)

Explain why ϕ is surjective, and use the rank theorem to establish that ϕ is injective, and hence an isomorphism.

(1 mark)

(b) Given that $Cl(4) \cong Mat(2, \boldsymbol{H})$, determine the structure of the algebra $Cl'(6)$ and hence determine the structure of $Cl(8)$.

(10 marks)

(c) Determine the structures of the Clifford algebras $Cl(16)$ and $Cl(24)$.

(5 marks)

You may use the following isomorphisms:

$$Cl'(n+2) \cong Cl(n) \otimes Mat(2, \boldsymbol{R})$$

$$Cl(n+2) \cong Cl'(n) \otimes \boldsymbol{H}$$

END OF QUESTIONS