[ENG18]

UNIVERSITY OF BOLTON

SCHOOL OF ENGINEERING

BSc (Hons) MATHEMATICS

SEMESTER 2 EXAMINATIONS 2021/22

GROUP THEORY

MODULE NO: MMA6005

Date: Thursday 19th May 2022 **Time: 14.00 - 16.15**

INSTRUCTIONS TO CANDIDATES:

1. **Please attempt all FOUR questions.**
2. **All questions carry equal marks.**
3. **Maximum marks for each part/question are shown in brackets.**

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1. (a) State, with reasons, whether or not each of the following groups is simple:

(i) $\boldsymbol{Z}_{11}$ (ii) $\boldsymbol{Z}_7 \times \boldsymbol{Z}_{11}$

(5 marks)

(b) Calculate the order of the symmetric group S_6 of permutations of a six element set.

Find the number of elements of type $(abc)(de)$ in S_6.

Find the centraliser of the permuation $(125)(34)$ in S_6.

Hence find the size of the conjugacy class of $(125)(34)$ in S_6.

(11 marks)

(c) Explain why the order of the general linear group $GL(3, \boldsymbol{Z}_p)$ is

$$(p^3 - 1)(p^3 - p)(p^3 - p^2)$$

Find a formula for the order of $SL(3, \boldsymbol{Z}_p)$, and calculate the order of $SL(3,5)$.

(9 marks)

2. (a) Find *two* composition series for the group $\boldsymbol{Z}_{28}$.

In each case, state the composition factors.

State, with reasons, whether or not $\boldsymbol{Z}_{28}$ is soluble.

(8 marks)

(b) Suppose that *M* and *N* are normal subgroups of a group *G*.

Let $MN = \{ab : a \in M, b \in N\}$.

Show that MN is a subgroup of *G*, and furthermore that it is a normal subgroup of *G*. (10 marks)

(c) Find a composition series for the symmetric group S_6, and state the composition factors.

Explain carefully whether or not S_6 is a soluble group.

(7 marks)

Please turn the page

3. (a) Find the minimal polynomial of $\alpha = \sqrt{1+\sqrt[4]{2}}$ over ***Q***.
State the degree of α over ***Q***.

(5 marks)

(b) Suppose that *E* is an extension field of *F*.

State what it means for *E* to be an *algebraic extension* of *F*, and for *E* to be a *finite extension*.

Show that if *E* is a finite extension of *F* then *E* is an algebraic extension of *F*.

(10 marks)

(c) List the 8th roots of unity. State which of these are primitive 8th roots of unity.

Let ω be a primitive 8th root of unity.

Find the minimal polynomial for ω and hence find a basis for the extension field $\boldsymbol{Q}(\omega)$ as a vector space over $\boldsymbol{Q}$.

(10 marks)

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4. (a) Using Eisenstein's criterion, show that the polynomial

$$f(x) = x^5 - 6x^3 - 21x + 12$$

is irreducible over $\boldsymbol{Q}$.

(4 marks)

(b) Let E be the splitting field of the polynomial $f(x)$ of part (a).

Let G be the Galois group $G = Gal(E/\boldsymbol{Q})$, and α a root of $f(x)$.

State the degree of α.

Show that if $\sigma \in Gal(E/\boldsymbol{Q})$ then $\sigma(\alpha)$ is also a root of $f(x)$.

(8 marks)

(c) Explain why the Galois group G of part (b) must have a subgroup of order 5.

(6 marks)

(d) Show that the cycle $(1\ 2\ 3\ 4\ 5)$ and the transposition $(1\ 2)$ generate the group S_5.

(7 marks)

END OF QUESTIONS