# UNIVERSITY OF BOLTON

# SCHOOL OF ENGINEERING

# B. Sc. (Hons) MATHEMATICS

# SEMESTER 1: EXAMINATION 2019/20

# COMPLEX VARIABLES

# MODULE NUMBER: MMA6006

**Date:** $16^{\text{th}}$ **January 2020** **Time: 10.00am – 12.15pm**

---

**INSTRUCTIONS TO CANDIDATES:**

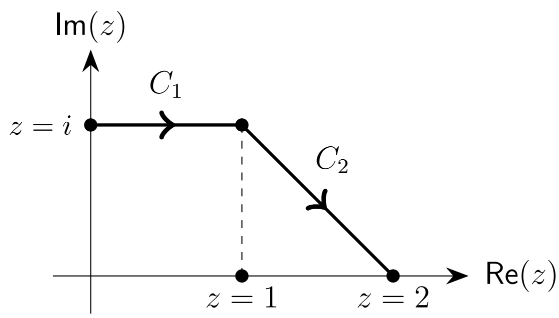
1. **Answer all FOUR questions.**
2. **Each question is worth 25 marks. The maximum marks possible for each part is shown in brackets.**
3. **The examination is closed-book.**
4. **The last two pages contain relevant definitions and results.**

---

PAST EXAMINATION PAPER

School of Engineering
B.Sc. Mathematics
Semester 1: Examination 2019/20
Complex Variables
MMA6006

1. (a) Consider the following piecewise contour $C = C_1 + C_2$:




Evaluate the integral $\displaystyle\int_C (2\,\overline{z} - z)\,dz$. (9 marks)

(b) Let $C$ denote the circle of radius $2$ centred at $z = 2$ traversed in an anti-clockwise direction starting from $z = 4$. Consider the complex function:

$$f(z) \;=\; \frac{5z+7}{z^2+2z-3}.$$

(i) Draw a sketch of the contour $C$ on an Argand diagram, indicating the starting position, orientation of the contour and the singularities of the function $f$. (6 marks)

(ii) Use the diagram from (i), Cauchy's theorem and Cauchy's integral formula to evaluate:

$$\oint_C f(z)\,dz.$$ (10 marks)

**PLEASE TURN THE PAGE**

2. (a) Consider the complex function $f$ defined by

$$f(z) = \frac{z^3 + 1}{z^3 + z^2}.$$

(i) Find and classify the *apparent* isolated singularities of $f$ arising from the zeros in the denominator. (4 marks)

(ii) Compute the residue of $f$ at each singularity and state, with reasons, whether or not any of the singularities are removable. (5 marks)

(b) Let $f$ be the complex function defined by

$$f(z) = \frac{2z}{(z-1)(z-3)}.$$

Find the Laurent series for $f$ on each of the *three* annular regions centred at $z = 0$ where $f$ is holomorphic. (16 marks)

3. (a) Let $C$ denote the circle of radius $2$ centred at the origin traversed in the anti-clockwise direction. Evaluate the following integral using the residue theorem:

$$\int_C \frac{z}{\cos(z)}\, dz$$

(5 marks)

(b) Show that:

$$\int_{x=-\infty}^{\infty} \frac{\cos(x)}{x^2+9}\, dx = \frac{\pi}{3e^3}$$

by evaluating a suitable contour integral taken over a semi-circular arc in the upper half plane centred at the origin. (14 marks)

(c) Use Rouche's theorem to show that the polynomial $2z^5 + 6z - 1$ has four roots in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$ and one real root in the interval $0 < x < 1$. (6 marks)

4. A string of unit length is clamped at one end, whereas the other oscillates freely with height $\sin(t)$ at time $t$. If $u(t,x)$ denotes the height of the string at time $t$ and position $x$, the motion of the string is determined as a solution to the initial-boundary value problem:

$$\begin{cases} u_{tt} &= u_{xx} & \\ u(0,x) &= u_t(0,x) \; = \; 0 & (0 < x < 1) \\ u(t,0) &= 0 & (t > 0) \\ u(t,1) &= \sin(t) & (t > 0). \end{cases}$$

(a) Use the method of Laplace transforms to show that the solution can be written as the Bromwich contour integral:

$$u(t,x) \quad = \quad \frac{1}{2\pi i} \int_C \frac{e^{st}}{1+s^2} \frac{\sinh(xs)}{\sinh(s)} \, ds$$

where $C$ is a vertical line in $\mathbb{C}$ such that all singularities of the integrand lie to the left of $C$. (16 marks)

(b) Use the residue theorem to evaluate the integral in (a) and show:

$$u(x,t) = \frac{\sin(x)}{\sin(1)} \sin(t) + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - 1} \sin(n\pi x) \sin(n\pi t).$$ (9 marks)

**END OF QUESTIONS**

**TURN THE PAGE FOR DEFINITIONS AND RESULTS SHEET**

School of Engineering
B.Sc. Mathematics
Semester 1: Examination 2019/20
Complex Variables
MMA6006

# Definitions and Results

**ML-estimate**: Given a domain $D \subseteq \mathbb{C}$, a continuous function $f : D \to \mathbb{C}$ and smooth curve $C : [a, b] \to D$, then

$$\left| \int_C f(z)\, dz \right| \leq ML$$

where $L$ is the length of $C$ and $M$ is the maximum value of the $f$ on $C$:

$$M = \max\{|f(z)| : z \in C\} = \max\{|(f \circ C)(t)| : t \in [a, b]\}$$

**Cauchy's Theorem**: Let $D \subset \mathbb{C}$ be a simply connected domain, $f : D \to \mathbb{C}$ be holomorphic in $D$ and $C$ a piecewise smooth curve. Then

$$\oint_C f(z)\, dz = 0.$$

**Cauchy's Integral Formula**: Let $f : D \to \mathbb{C}$ be holomorphic in a simply connected domain $D$ and $C$ denote a simple, piecewise smooth, closed curve in $D$ with counter-clockwise orientation. Then

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C. \end{cases}$$

If $z_0$ is on $C$, then the integral is improper and may not even exist.

**Cauchy's Integral Formula for Derivatives**: Let $f : D \to \mathbb{C}$ be holomorphic in a simply connected domain $D$ and $C$ denote a simple, piecewise smooth, closed curve in $D$ with counter-clockwise orientation. Then for any point $z_0$ in the interior of $C$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

**Residues**: The coefficient $a_{-1}$ of $1/(z_{z0})$ in the Laurent series of a function $f$ about $z = z_0$ is called the residue of $f$. If $f$ has a pole of order $n$ at $z = z_0$ it can be computed as

$$\operatorname*{Res}_{z=z_0} f(z) \;=\; \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z).$$

**Residue Theorem**: Let $C$ be a simple, closed, piecewise smooth curve and $f : D \to \mathbb{C}$ be holomorphic in $D \subset \mathbb{C}$ and on $C$ except at a finite number of isolated singularities $\{z_1, z_2, \ldots, z_n\}$ lying interior to $C$. Then:

$$\oint_C f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

**Rouché's Theorem**: Let $f(z), g(z)$ be holomorphic in $D \subset \mathbb{C}$ and let $C$ be a simple closed contour in $D$ not passing through any zeros of $f$ or $f+g$. Assume $|f(z)| > |g(z)|$ for $z$ on $C$, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros (including multiplicities) inside $C$.

**Laplace Transforms**: The *Laplace transform* of a complex function $f : [0,\infty) \to \mathbb{C}$ is defined by:

$$F(s) \;\equiv\; \mathcal{L}\{\, f(t) \,\} \;=\; \int_{t=0}^{\infty} f(t)e^{-st}\,dt \;=\; \lim_{M\to\infty} \int_{t=0}^{M} f(t)e^{-st}\,dt$$

in terms of the complex parameter $s = x + iy$. If $f$ is piecewise continuous and of exponential order $\alpha$ then the integral exists for all Re$(s) > \alpha$. The *inverse Laplace transform* $\mathcal{L}^{-1}\{\, F(s) \,\}$ is given by the Bromwich contour integral:

$$f(t) \;=\; \mathcal{L}^{-1}\{\, F(s) \,\} \;=\; \frac{1}{2\pi i} \int_C F(s)e^{st}\,ds$$

where $C$ is a vertical contour in $\mathbb{C}$ parametrized by $C(t) = \alpha + it$ $(t \in \mathbb{R})$ such that all singularities of the integrand lie to the left of $C$. If $F(s) = \mathcal{L}\{\, f(t) \,\}$ has isolated singularities at $\{s_1, \ldots, s_n\}$ in the half-plane defined by Re$(s) < \alpha$ and $F(s) \to 0$ as $|s| \to \infty$:

$$f(t) \;=\; \mathcal{L}^{-1}\{\, F(s) \,\} \;=\; \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k} F(s)e^{st}.$$

# END OF PAPER